Journal of Fuzzy Extension and Applications

www.journal-fea.com

J. Fuzzy. Ext. Appl. Vol. 6, No. 3 (2024) 555–571.

Paper Type: Research Paper

Soft Optimal Functions and Soft Semi-Hausdorff Spaces

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Citation:

Received: 05-12-2024 Revised: 22-02-2025 Accepted: 27-04-2025	Al-Ghour, S. (2024). Soft optimal functions and soft semi-Hausdorff spaces. *Journal of Fuzzy Extension and Applications,6*(3), 555-571

Abstract

This paper presents and investigates the concept of soft optimal maps within the framework of soft topological spaces. A soft map is called soft optimal if it preserves soft disjointness for soft open sets in the soft structure. The link between soft injectivity and soft optimality is investigated, revealing that every soft injective map is soft optimal, but the converse is not true. Some correlations between this class of soft optimal maps and its analogs in general topology are given. The paper further shows that soft optimality is independent of all forms of soft continuity, soft openness, and soft closedness. Furthermore, the impact of soft optimality on soft semi-Hausdorff and soft Hausdorff properties is considered with concrete theorems showing how soft optimality affects the soft semi-Hausdorff state in soft topological spaces under various soft continuity assumptions. Finally, we show that the soft optimal map, which is soft s-open and soft pre-continuous, will preserve soft disjointness for the class of soft semi-open sets.

Keywords: Soft Hausdorff, Soft semi-Hausdorff, Soft pre-continuous function, Soft semi-continuous function, Generated soft topology.

1|Introduction and Preliminaries

Since its introduction by Molodtsov in 1999 [1], soft set theory has attracted much attention due to its adaptability and ability to represent uncertainty in a variety of contexts. Soft set theory is very flexible because it allows the use of parameters to indicate uncertainty, unlike traditional set theory, which deals with binary relationships.

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https://doi.org/10.22105/jfea.2025.492313.1728

Among its main advantages over other uncertainty theories, especially fuzzy set theory and approximate set theory, is its ease of use and simplicity. Soft sets do not require the complex operations or membership maps typically required in fuzzy sets [2]. Soft set theory's ability to accommodate imprecise and fuzzy data makes it a valuable tool for decision-making when precise information is unavailable. Its main advantage over other uncertainty theories, such as fuzzy set theory or approximate set theory, is its ease of use and simplicity: Soft sets do not require complex operations, and membership maps are often required for fuzzy sets. As research continues, so does its ability to solve complex world problems. Real-world areas such as medical diagnosis and pattern recognition continue to expand. In addition, soft set theory can be extended to multi-criteria decision-making, providing a useful framework for evaluating different factors in uncertain environments. After all, the combination of simplicity, application, and effectiveness makes it a promising theory for modeling uncertainty. Using soft sets, several mathematical structures appeared, and many research papers appeared in those structures [3–22].

Shabir and Naz [23] started the research of soft topology, which has brought important ideas that go beyond conventional topological ideas to deal with ambiguity and uncertainty. One of the fundamental concepts is soft compactness which permits the analysis of compactness in situations where membership is parameterized by extending the traditional concept of compact sets to soft topological spaces. Soft connectedness expansion of the concept of connectedness by utilizing the adaptability of soft sets is another noteworthy addition. This makes it possible to investigate connectedness in regions without ambiguous borders. For managing maps across soft topological spaces, the conventional concept of continuous maps has been expanded to include soft continuity. Moreover, to define concepts like convergence and continuity in a soft context the idea of a soft metric was proposed to measure distances in soft topological spaces. Further, by defining how different soft points and soft sets can be separated in soft spaces some authors introduced soft separation axioms which are generalizations of classical separation axioms that contributed to applications in fuzzy logic decisions and mathematical modeling these ideas have significantly expanded the field of topological research. [24–35] provides some examples of traditional topological concepts that have been developed and expanded in soft set environments. After the idea of soft mapping with details was first presented in [36], soft continuity for soft mappings was established in [37]. Soft continuity notions have been the focus of research ever since (see some of the recent [38–41]).

In this paper, the concept of soft optimal maps, a new class of maps within the context of soft set theory, is introduced. These maps use the flexibility of soft sets, which enable the representation of ambiguity and uncertainty, to expand on traditional ideas in mathematical logic and topology. To extend the scope of soft set theory, we investigate the fundamental properties of soft optimal maps and show that they belong to a class distinct from the well-known soft injective maps. An important contribution is to find connections between soft optimal maps and their broad topological counterparts, which sheds light on their topological counterparts. We also investigate how well-known soft separation axioms, such as those for the separation of points and sets, are preserved in soft optimal maps under certain additional conditions. This work adds to the growing body of research aimed at bridging the gap between classical topological spaces and soft set theory. The results of the paper open up new directions for research at the intersection of mathematical logic, soft sets, and topology. Thus, taking up conclusions, these soft optimal maps open an utterly new field for research on the soft topological spaces meaning that the soft set theory has allowed for new and potential use in the branches that rely almost entirely, upon the very formulation of problems related to uncertainties and imprecise.

"Assume that Z is a non-empty set and $\mathcal{T}$ is a set of parameters. A soft set over Z relative to $\mathcal{T}$ is a map $K : \mathcal{T} \longrightarrow \mathcal{P}(Z)$. $SS(Z,\mathcal{T})$ denotes the family of all soft sets over Z relative to $\mathcal{T}$. The null soft set and the absolute soft set are denoted by $0_{\mathcal{T}}$ and $1_{\mathcal{T}}$, respectively. Let $K \in SS(Z,\mathcal{T})$. If $K(a) = N$ for all $a \in \mathcal{T}$, then K is denoted by C_N. If $K(a) = N$ and $K(b) = \emptyset$ for all $b \in \mathcal{T} - \{a\}$, then K is denoted by a_N. If $K(a) = \{z\}$ and $K(b) = \emptyset$ for all $b \in \mathcal{T} - \{a\}$, then K is called a soft point over Z relative to $\mathcal{T}$ and denoted by a_z. $SP(Z,\mathcal{T})$ denotes the family of all soft points over Z relative to $\mathcal{T}$. If $K \in SS(Z,\mathcal{T})$ and $a_z \in SP(Z,\mathcal{T})$, then a_z is said to belong to K (notation: $a_z \widetilde{\in} K$) if $z \in K(a)$. Let $SS(Z,\mathcal{T})$ and $SS(Y,\mathcal{S})$ be two families of soft sets, and $n : Z \longrightarrow Y$, $u : \mathcal{T} \longrightarrow S$ be two maps. Then a soft map $f_{nu} : SS(Z,\mathcal{T}) \longrightarrow SS(Y,\mathcal{S})$ is defined as follows: For each $H \in SS(Z,\mathcal{T})$ and $K \in SS(Y,\mathcal{S})$, $(f_{nu}(H))(b) = \emptyset$ if $u^{-1}(b) = \emptyset$, $(f_{nu}(H))(b) = \cup_{a \in u^{-1}(b)} n(H(a))$ if $u^{-1}(b) \neq \emptyset$, and $\left(f_{nu}^{-1}(K)\right)(a) = n^{-1}(K(u(a)))$ for all $a \in \mathcal{T}$. The soft map $f_{nu} : SS(Z,\mathcal{T}) \longrightarrow SS(Y,\mathcal{S})$ is soft injective (resp. soft surjective, soft bijective) if $n : Z \longrightarrow Y$ and $u : \mathcal{T} \longrightarrow S$ are both injective (resp. surjective, bijective). The triplet $(Z,\eta,\mathcal{T})$, where $\eta \subseteq SS(Z,\mathcal{T})$, is known as a soft topological space if

$\{0_{\mathcal{T}}, 1_{\mathcal{T}}\} \subseteq \eta$, η is closed under finite soft intersection and arbitrary soft union. If $(Z, \eta, \mathcal{T})$ is a soft topological space, then the members of η are called soft open sets in $(Z, \eta, \mathcal{T})$, while their soft complements are known as soft closed sets $(Z, \eta, \mathcal{T})$. η^c denotes the family of all soft closed sets in a soft topological space $(Z, \eta, \mathcal{T})$. Let $(Z, \eta, \mathcal{T})$ be a soft topological space, and let $K \in SS(Z, \mathcal{T})$. Then the soft interior and the soft closure of K in $SS(Z, \mathcal{T})$ will be denoted by $Int_\eta(K)$ and $Clo_\eta(K)$, respectively, and will be defined as follows:

$Int_\eta(K) = \widetilde{\cup}\left\{H : H \in \eta \text{ and } H \widetilde{\subseteq} K\right\}$ and

$Clo_\eta(K) = \widetilde{\cap}\left\{H : H \in \eta^c \text{ and } K \widetilde{\subseteq} H\right\}$."

We will use concepts and terminology from [42, 43] throughout this work for clarification. The abbreviations TS and STS stand for topological space and soft topological space, respectively.

The following definitions will be used in the sequel:

Definition 1. [44] A map $g : (Z, \Psi) \longrightarrow (Y, \Gamma)$ is called "optimal" if for any $U, V \in \Psi$, $U \cap V = \emptyset$ implies $g(U) \cap g(V) = \emptyset$.

Every injective map is optimal. In contrast, if $Z = \mathbb{R}$, $\Psi = \{\emptyset, \mathbb{R}, (-\infty, 0), [0, \infty)\}$, and $g : (\mathbb{R}, \Psi) \longrightarrow (\mathbb{R}, \Psi)$ defined by $g(x) = 1$ if $x \in (-\infty, 0)$ and $g(x) = -1$ if $x \in [0, \infty)$, then g is optimal but not injective.

Definition 2. (a) Let $(Z, \eta, \mathcal{T})$ be a STS and let $H \in SS(Z, \mathcal{T})$. Then

(a) H is a "soft semi-open" [45, 46] (resp. "soft pre-open" [47], "soft α-open" [48], "soft regular-open" [49]) set in $(Z, \eta, \mathcal{T})$ if $H \widetilde{\subseteq} Clo_\eta(Int_\eta(H))$ (resp. $H \widetilde{\subseteq} Int_\eta(Clo_\eta(H))$, $H \widetilde{\subseteq} Int_\eta(Clo_\eta(Int_\eta(H)))$, $H = Int_\eta(Clo_\eta(H))$). The family of all soft semi-open sets (resp. soft pre-open, soft α-open sets, soft regular-open sets) in $(Z, \eta, \mathcal{T})$ will be denoted by $SO(\eta)$ (resp. $PO(\eta)$, $\alpha(\eta)$, $RO(\eta)$).

(b) [45] H is called a "soft semi-closed set" in $(Z, \eta, \mathcal{T})$ if $1_{\mathcal{T}} - H \in SO(\eta)$. The family of all soft semi-closed sets in $(Z, \eta, \mathcal{T})$ will be denoted by $SC(\eta)$.

(c) [45] The "soft semi-closure" of H in $(Z, \eta, \mathcal{T})$ is denoted by $sCl_\eta(H)$ and defined by

$$sClo_\eta(H) = \widetilde{\cap}\left\{K : K \in SC(\eta) \text{ and } H \widetilde{\subseteq} K\right\}.$$

Definition 3. A STS $(Z, \eta, \mathcal{T})$ is called "soft Hausdorff" [50] (resp. "soft semi-Hausdorff" [51]) if for any $a_x, b_y \in SP(Z, \mathcal{T})$ such that $a_x \neq b_y$, we find $K, H \in \eta$ (resp. $K, H \in SO(\eta)$) such that $a_x \widetilde{\in} K$, $b_y \widetilde{\in} H$, and $K \widetilde{\cap} H = 0_{\mathcal{T}}$.

Definition 4. A soft map $f_{nu} : (Z, \eta, \mathcal{T}) \longrightarrow (Y, \lambda, \mathcal{S})$ is said to be

(a) "soft semi-continuous" [52] (resp. "soft pre-continuous" [48], "soft α-continuous" [48]) if for each $H \in \lambda$, $f_{nu}^{-1}(H) \in SO(\eta)$ (resp. $f_{nu}^{-1}(H) \in PO(\eta)$, $f_{nu}^{-1}(H) \in \alpha(\eta)$).

(b) "soft semi-open" [45] (resp. "soft pre-open" [48], "soft α-open" [48]) if for each $K \in \eta$, $f_{nu}(K) \in SO(\lambda)$ (resp. $f_{nu}(K) \in PO(\lambda)$, $f_{nu}(K) \in \alpha(\lambda)$).

(c) "soft almost-open" (resp. "soft s-open", "soft closed-open") if $f_{nu}(K) \in \lambda$ for each $K \in RO(\eta)$ (resp. $K \in SO(\eta)$, $K \in \eta^c$).

(d) "soft pre-semi-closed" if for each $K \in SC(\eta)$, $f_{nu}(K) \in SC(\lambda)$.

Theorem 1. [42] Let $\{(Z, \eta_a) : a \in \mathcal{T}\}$ be a collection of TSs and let $\oplus_{a \in \mathcal{T}} \eta_a = \{G \in SS(Z, \mathcal{T}) : G(a) \in \eta_a \text{ for all } a \in \mathcal{T}\}$. Then $(Z, \oplus_{a \in \mathcal{T}} \eta_a, \mathcal{T})$ is a STS.

In Theorem 1, if $\eta_a = \Im$ for all $a \in \mathcal{T}$, then we will denote $\oplus_{a\in\mathcal{T}}\eta_a$ by $\tau(\Im)$.

2|Results

Definition 5. A soft map $f_{nu} : (Z, \eta, \mathcal{T}) \longrightarrow (Y, \lambda, \mathcal{S})$ is called "soft optimal" if for any $G, H \in \eta$, $G\widetilde{\cap}H = 0_{\mathcal{T}}$ implies $f_{nu}(G)\widetilde{\cap}f_{nu}(H) = 0_{\mathcal{S}}$.

The following result shows that soft injective maps are a subclass of optimal maps:

Theorem 2. Every soft injective map is soft optimal.

Proof. Let $f_{nu} : (Z, \eta, \mathcal{T}) \longrightarrow (Y, \lambda, \mathcal{S})$ be soft injective. Let $G\widetilde{\cap}H = 0_{\mathcal{T}}$. Then $f_{nu}\left(G\widetilde{\cap}H\right) = f_{nu}(0_{\mathcal{T}}) = 0_{\mathcal{S}}$. On the other hand, since f_{nu} is soft injective, then $f_{nu}\left(G\widetilde{\cap}H\right) = f_{nu}(G)\widetilde{\cap}f_{nu}(H)$. Therefore, $f_{nu}(G)\widetilde{\cap}f_{nu}(H) = 0_{\mathcal{S}}$. It follows that f_{nu} is soft optimal.

The following example demonstrates that Theorem 2's converse need not be true:

Example 1. Let $Z = \mathbb{R}$, $\mathcal{T} = \{s, t\}$, and $\eta = \{0_{\mathcal{T}}, 1_{\mathcal{T}}, C_{\mathbb{Q}}, C_{\mathbb{R}-\mathbb{Q}}\}$. Define $n : Z \longrightarrow Z$ and $u : \mathcal{T} \longrightarrow \mathcal{T}$ by $n(x) = 1$ if $x \in \mathbb{Q}$, $n(x) = -1$ if $x \in \mathbb{R} - \mathbb{Q}$, $u(s) = s$, and $u(t) = t$. Then $f_{nu} : (Z, \eta, \mathcal{T}) \longrightarrow (Z, \eta, \mathcal{T})$ is soft optimal but not soft injective.

The soft optimality of soft maps and their classical topological analogs are correlated in the following two results:

Theorem 3. Let $\{(Z, \eta_a) : a \in \mathcal{T}\}$ and $\{(Y, \lambda_b) : b \in \mathcal{S}\}$ be two collections of TSs. Let $n : Z \longrightarrow Y$ and $u : \mathcal{T} \longrightarrow \mathcal{S}$ be maps with u being injective. Then $f_{nu} : (Z, \oplus_{a\in\mathcal{T}}\eta_a, \mathcal{T}) \longrightarrow (Y, \oplus_{b\in\mathcal{S}}\lambda_b, \mathcal{S})$ is soft optimal iff $n : (Z, \eta_a) \longrightarrow \left(Y, \lambda_{u(a)}\right)$ is optimal for all $a \in \mathcal{T}$.

Proof. *Necessity.* Let $f_{nu} : (Z, \oplus_{a\in\mathcal{T}}\eta_a, \mathcal{T}) \longrightarrow (Y, \oplus_{b\in\mathcal{S}}\lambda_b, \mathcal{S})$ be soft optimal. Let $d \in \mathcal{T}$. Let $O, W \in \eta_d$ such that $O \cap W = \emptyset$. Then $d_O, d_W \in \oplus_{a\in\mathcal{T}}\eta_a$ and $d_O\widetilde{\cap}d_W = d_{O\cap W} = d_{\emptyset} = 0_{\mathcal{T}}$. Since f_{nu} is soft optimal, $f_{nu}(d_O)\widetilde{\cap}f_{nu}(d_W) = 0_{\mathcal{S}}$. Since $f_{nu}(d_O) = (u(d))_{n(O)}$ and $f_{nu}(d_O) = (u(d))_{n(W)}$, then $(u(d))_{n(O)}\widetilde{\cap}(u(d))_{n(\ W)} = (u(d))_{n(O)\cap n(W)} = 0_{\mathcal{S}}$. Hence, $n(O) \cap n(W) = \emptyset$. Therefore, $n : (Z, \eta_d) \longrightarrow \left(Y, \lambda_{u(d)}\right)$ is optimal.

Sufficiency. Let $n : (Z, \eta_a) \longrightarrow \left(Y, \lambda_{u(a)}\right)$ be optimal for all $a \in \mathcal{T}$. Suppose to the contrary that we find $G, H \in \eta$ such that $G\widetilde{\cap}H = 0_{\mathcal{T}}$ and $f_{nu}(G)\widetilde{\cap}f_{nu}(H) \neq 0_{\mathcal{S}}$. Choose $b_y\widetilde{\in}f_{nu}(G)\widetilde{\cap}f_{nu}(H)$. Then we find $a_x\widetilde{\in}G$ and $d_z\widetilde{\in}H$ such that $b_y = f_{nu}(a_x) = (u(a))_{n(x)}$ and $b_y = f_{nu}(d_z) = (u(d))_{n(z)}$. Thus, we have $(u(a))_{n(x)} = (u(d))_{n(z)}$; hence $u(a) = u(d)$ and $n(x) = n(z)$. Since u is injective, then $a = d$. Since $a_x\widetilde{\in}G$ and $d_z\widetilde{\in}H$, then $x \in G(a)$ and $z \in H(d) = H(a)$. So, we have $n(x) = n(z) \in n(G(a)) \cap n(H(a))$. Since $G\widetilde{\cap}H = 0_{\mathcal{T}}$, then $G(a) \cap H(a) = \emptyset$. Since $n : (Z, \eta_a) \longrightarrow \left(Y, \lambda_{u(a)}\right)$ is optimal, $G(a), H(a) \in \eta_a$, and $G(a) \cap H(a) = \emptyset$, then $n(G(a)) \cap n(H(a)) = \emptyset$, which is a contradiction.

Corollary 1. Let $u : (Z, \aleph) \longrightarrow (Y, \Im)$ and $u : \mathcal{T} \longrightarrow \mathcal{S}$ be two maps where u is injective. Then $u : (Z, \aleph) \longrightarrow (Y, \Im)$ is optimal iff $f_{nu} : (Z, \tau(\aleph), \mathcal{T}) \longrightarrow (Y, \tau(\Im), \mathcal{S})$ is soft optimal.

Proof. For every $a \in \mathcal{T}$ and $b \in \mathcal{S}$, let $\eta_a = \aleph$ and $\lambda_b = \Im$. Then $\tau(\aleph) = \oplus_{a\in\mathcal{T}}\eta_a$ and $\tau(\Im) = \oplus_{b\in\mathcal{S}}\lambda_b$. Theorem 3 ends the proof.

The following three examples show that soft optimality is independent of all soft continuity, soft openness, and soft closedness.

Example 2. Let $Z = \mathbb{R}$, $\mathcal{T} = \{s, t\}$, $\aleph = \{\emptyset, Z\} \cup \{(-\infty, r) : r \in \mathbb{R}\}$, and $\Im = \{\emptyset, Z\} \cup \{(r, \infty) : r \in \mathbb{R}\}$. Define $n : Z \longrightarrow Z$ and $u : \mathcal{T} \longrightarrow \mathcal{T}$ by $n(x) = -x$ for all $x \in Z$, $u(s) = s$, and $u(t) = t$. Then

$f_{nu}:(Z,\tau(\aleph),\mathcal{T})\longrightarrow(Z,\tau(\Im),\mathcal{T})$ is soft optimal, but it is not soft continuous, not soft open, and not soft closed.

Example 3. Let $Z=\mathbb{R}$, $\mathcal{T}=\{a,b\}$, $\eta=SS(Z,\mathcal{T})$, and $\lambda=\{0_{\mathcal{T}},1_{\mathcal{T}},a_1\}$. Define $n:Z\longrightarrow Z$ and $u:\mathcal{T}\longrightarrow\mathcal{T}$ by $n(z)=1$ for all $z\in Z$, $u(a)=a$, and $u(b)=a$. Then $f_{nu}:(Z,\eta,\mathcal{T})\longrightarrow(Z,\lambda,\mathcal{T})$ is soft continuous and soft open, but it is not soft optimal.

Example 4. Let $Z=\mathbb{R}$, $\mathcal{T}=\{a,b\}$, $\eta=SS(Z,\mathcal{T})$, and $\lambda=\{0_{\mathcal{T}},1_{\mathcal{T}},1_{\mathcal{T}}-a_1\}$. Define $n:Z\longrightarrow Z$ and $u:\mathcal{T}\longrightarrow\mathcal{T}$ by $n(z)=1$ for all $z\in Z$, $u(a)=a$, and $u(b)=a$. Then $f_{nu}:(Z,\eta,\mathcal{T})\longrightarrow(Z,\lambda,\mathcal{T})$ is soft closed but it is not soft optimal.

The sequel will be dependent on the following lemma:

Lemma 1. Let $(Z,\eta,\mathcal{T})$ be a STS and let $K\in SO(\eta)$. Then $Clo_{\alpha(\eta)}(K)=Clo_{\eta}(K)$.

Proof. Since $\eta\subseteq\alpha(\eta)$, then $Clo_{\alpha(\eta)}(K)\widetilde{\subseteq}Clo_{\eta}(K)$. To see that $Clo_{\eta}(K)\widetilde{\subseteq}$

$Clo_{\alpha(\eta)}(K)$, suppose to the contrary that we find $a_z\widetilde{\in}Clo_{\eta}(K)-Clo_{\alpha(\eta)}(K)$. Since $a_z\widetilde{\notin}Clo_{\alpha(\eta)}(K)$, there is $G\in\alpha(\eta)$ such that $a_z\widetilde{\in}G$ while $G\widetilde{\cap}K=0_{\mathcal{T}}$, which implies that $Int_{\eta}(K)\widetilde{\cap}Int_{\eta}(G)=0_{\mathcal{T}}$, which in turn implies that $Int_{\eta}(K)\widetilde{\cap}Clo_{\eta}(Int_{\eta}(G))=0_{\mathcal{T}}$ and hence

$$Int_{\eta}(K)\widetilde{\cap}Int_{\eta}(Clo_{\eta}(Int_{\eta}(G)))=0_{\mathcal{T}}.$$

Thus, $Clo_{\eta}(Int_{\eta}(K))\widetilde{\cap}Int_{\eta}(Clo_{\eta}(Int_{\eta}(G)))=0_{\mathcal{T}}$. Since $K\in SO(\eta)$, then $K\widetilde{\subseteq}Clo_{\eta}(Int_{\eta}(K))$. Therefore, $K\widetilde{\cap}Int_{\eta}(Clo_{\eta}(Int_{\eta}(G)))=0_{\mathcal{T}}$. On the other hand, since $a_z\widetilde{\in}Clo_{\eta}(K)$ and $a_z\widetilde{\in}Int_{\eta}(Clo_{\eta}(Int_{\eta}(G)))\in\eta$, then $K\widetilde{\cap}Int_{\eta}(Clo_{\eta}(Int_{\eta}(G)))\neq 0_{\mathcal{T}}$, a contradiction.

Theorems 4, 5, and 6 introduce three preservation theorems of soft semi-Hausdorff spaces via soft optimality.

Theorem 4. Let $f_{nu}:(Z,\eta,\mathcal{T})\longrightarrow(Y,\lambda,\mathcal{S})$ be soft surjective, soft pre-continuous, soft α-open, and soft optimal. If $(Z,\eta,\mathcal{T})$ is soft semi-Hausdorff, then $(Y,\lambda,\mathcal{S})$ is soft semi-Hausdorff.

Proof. Let $b_y,d_z\in SP(Y,\mathcal{S})$ such that $b_y\neq d_z$. Since f_{nu} is soft surjective, then we find $a_x,e_w\in SP(Z,\mathcal{T})$ such that $f_{nu}(a_x)=b_y$ and $f_{nu}(e_w)=d_z$. Since $b_y\neq d_z$, then $a_x\neq e_w$. Since $(Z,\eta,\mathcal{T})$ is soft semi-Hausdorff, we find $G,H\in SO(\eta)$ such that $a_x\widetilde{\in}G$, $e_w\widetilde{\in}H$, and $G\widetilde{\cap}H=0_{\mathcal{T}}$. As $G,H\in SO(\eta)$, we find $L,M\in\eta$ such that $L\widetilde{\subseteq}G\widetilde{\subseteq}Clo_{\eta}(L)$ and $M\widetilde{\subseteq}H\widetilde{\subseteq}Clo_{\eta}(M)$. Since f_{nu} is soft pre-continuous, then by Theorem 18 of [53], $f_{nu}(L)\widetilde{\subseteq}f_{nu}(G)\widetilde{\subseteq}f_{nu}(Clo_{\eta}(L))\widetilde{\subseteq}Clo_{\lambda}(f_{nu}(L))$ and

$f_{nu}(M)\widetilde{\subseteq}f_{nu}(H)\widetilde{\subseteq}f_{nu}(Clo_{\eta}(M))\widetilde{\subseteq}Clo_{\lambda}(f_{nu}(M))$.

Moreover, as f_{nu} is soft α-open, $f_{nu}(L),f_{nu}(M)\in\alpha(\lambda)\subseteq SO(\lambda)$. By Theorem 3.3 of [45], we conclude that $f_{nu}(G),f_{nu}(H)\in SO(\lambda)$. Obviously, $b_y\widetilde{\in}f_{nu}(G)\widetilde{\subseteq}Clo_{\lambda}(f_{nu}(G))=Clo_{\lambda}(f_{nu}(L))$ and $d_z\widetilde{\in}f_{nu}(H)\widetilde{\subseteq}Clo_{\lambda}(f_{nu}(H))=Clo_{\lambda}(f_{nu}(M))$. By Lemma 1, $Clo_{\lambda}(f_{nu}(L))=Clo_{\alpha(\lambda)}(f_{nu}(L))$ and $Clo_{\lambda}(f_{nu}(M))=Clo_{\alpha(\lambda)}(f_{nu}(M))$. Let $U=b_y\widetilde{\cup}f_{nu}(L)$ and $V=d_z\widetilde{\cup}f_{nu}(M)$. Then

$f_{nu}(L)\widetilde{\subseteq}U\widetilde{\subseteq}Clo_{\lambda}(f_{nu}(L))$ and $f_{nu}(M)\widetilde{\subseteq}V\widetilde{\subseteq}Clo_{\lambda}(f_{nu}(M))$, and so, $U,V\in SO(\lambda)$.

Claim. 1. $f_{nu}(L)\widetilde{\cap}f_{nu}(M)=0_{\mathcal{S}}$.

2. $b_y\widetilde{\notin}f_{nu}(M)$.

3. $d_z\widetilde{\notin}f_{nu}(L)$.

Proof of Claim. 1. Since $L,M\in\eta$ and $L\widetilde{\cap}M\widetilde{\subseteq}G\widetilde{\cap}H=0_{\mathcal{T}}$, then by soft optimality of f_{nu}, $f_{nu}(L)\widetilde{\cap}f_{nu}(M)=0_{\mathcal{S}}$.

2. Suppose to the contrary that $b_y \widetilde{\in} f_{nu}(M)$. Since $f_{nu}(M) \in \alpha(\lambda)$ and $b_y \widetilde{\in} Clo_{\alpha(\lambda)}(f_{nu}(L))$, then $f_{nu}(L) \widetilde{\cap} f_{nu}(M) = 0_{\mathcal{S}}$, which contradicts 1.

3. Suppose to the contrary that $d_z \widetilde{\notin} f_{nu}(L)$. Since $f_{nu}(L) \in \alpha(\lambda)$ and $d_z \widetilde{\in} Clo_{\alpha(\lambda)}(f_{nu}(M))$, then $f_{nu}(L) \widetilde{\cap} f_{nu}(M) = 0_{\mathcal{S}}$, which contradicts 1.

By the above claim,

$$\begin{array}{cc} U \widetilde{\cap} V & = \\ \left(b_y \widetilde{\cup} f_{nu}(L)\right) \widetilde{\cap} \left(d_z \widetilde{\cup} f_{nu}(M)\right) & = \\ \left(b_y \widetilde{\cap} f_{nu}(M)\right) \widetilde{\cup} \left(d_z \widetilde{\cap} f_{nu}(L)\right) \widetilde{\cup} \left(f_{nu}(L) \widetilde{\cap} f_{nu}(M)\right) & = \\ 0_{\mathcal{S}}. & \end{array}$$

Therefore, we have $b_y \widetilde{\in} U \in SO(\lambda)$, $d_z \widetilde{\in} V \in SO(\lambda)$, and $U \widetilde{\cap} V = 0_{\mathcal{S}}$. Hence, $(Y, \lambda, \mathcal{S})$ is soft semi-Hausdorff.

The following question is natural:

Is it true that soft pre-continuous soft α-open maps are soft optimal?

The following example answers the above question:

Example 5. Let $Z = \{1, 2, 3\}$, $Y = \{4\}$, $\mathcal{T} = \{a\}$, $\eta = \{a_U : U \subseteq Z\}$, and $\lambda = \{0_{\mathcal{T}}, 1_{\mathcal{T}}\}$. Consider the constant maps $n : Z \longrightarrow Y$ and $u : \mathcal{T} \longrightarrow \mathcal{T}$. Then $f_{nu} : (Z, \eta, \mathcal{T}) \longrightarrow (Y, \lambda, \mathcal{T})$ is soft pre-continuous and soft α-open but not soft optimal.

Lemma 2. Let $(Z, \eta, \mathcal{T})$ be a STS and let $K \in SO(\eta)$. Then for any $R \in SS(Z, \mathcal{T})$,

$$K \widetilde{\cap} sClo_{\eta}(R) \widetilde{\subseteq} Clo_{\eta}\left(K \widetilde{\cap} R\right).$$

Proof. Let $a_z \widetilde{\in} K \widetilde{\cap} sClo_{\eta}(R)$ and let $G \in \eta$ such that $a_z \widetilde{\in} G$. Since $a_z \widetilde{\in} K \widetilde{\cap} G \in SO(\eta)$ and $a_z \widetilde{\in} sClo_{\eta}(R)$, then $\left(K \widetilde{\cap} G\right) \widetilde{\cap} R = G \widetilde{\cap} \left(K \widetilde{\cap} R\right) \neq 0_{\mathcal{T}}$. Hence, $a_z \widetilde{\in} Clo_{\eta}\left(K \widetilde{\cap} R\right)$.

Theorem 5. Let $f_{nu} : (Z, \eta, \mathcal{T}) \longrightarrow (Y, \lambda, \mathcal{S})$ be soft surjective, soft α-continuous, soft semi-open, soft closed-open, and soft optimal. If $(Z, \eta, \mathcal{T})$ is soft semi-Hausdorff, then $(Y, \lambda, \mathcal{S})$ is soft semi-Hausdorff.

Proof. Let $b_y, d_z \in SP(Y, \mathcal{S})$ such that $b_y \neq d_z$. Since f_{nu} is soft surjective, then we find $a_x, e_w \in SP(Z, \mathcal{T})$ such that $f_{nu}(a_x) = b_y$ and $f_{nu}(e_w) = d_z$. Since $b_y \neq d_z$, then $a_x \neq e_w$. Since $(Z, \eta, \mathcal{T})$ is soft semi-Hausdorff, we find $G, H \in SO(\eta)$ such that $a_x \widetilde{\in} G$, $e_w \widetilde{\in} H$, and $G \widetilde{\cap} H = 0_{\mathcal{T}}$. As $G, H \in SO(\eta)$, we find $L, M \in \eta$ such that $L \widetilde{\subseteq} G \widetilde{\subseteq} Clo_{\eta}(L)$ and $M \widetilde{\subseteq} H \widetilde{\subseteq} Clo_{\eta}(M)$. By Lemma 1, $Clo_{\eta}(L) = Clo_{\alpha(\eta)}(L)$ and $Clo_{\eta}(M) = Clo_{\alpha(\eta)}(M)$. Since $f_{nu} : (Z, \eta, \mathcal{T}) \longrightarrow (Y, \lambda, \mathcal{S})$ is soft α-continuous, then $f_{nu} : (Z, \alpha(\eta), \mathcal{T}) \longrightarrow (Y, \lambda, \mathcal{S})$ is soft continuous; hence

$f_{nu}\left(Clo_{\alpha(\eta)}(L)\right) \widetilde{\subseteq} Clo_{\lambda}(f_{nu}(L))$ and $f_{nu}\left(Clo_{\alpha(\eta)}(M)\right) \widetilde{\subseteq} Clo_{\lambda}(f_{nu}(M))$.

So, we have

$f_{nu}(L) \widetilde{\subseteq} f_{nu}(G) \widetilde{\subseteq} f_{nu}(Clo_{\eta}(L)) = f_{nu}\left(Clo_{\alpha(\eta)}(L)\right) \widetilde{\subseteq} Clo_{\lambda}(f_{nu}(L))$

and

$f_{nu}(M) \widetilde{\subseteq} f_{nu}(H) \widetilde{\subseteq} f_{nu}(Clo_{\eta}(M)) = f_{nu}\left(Clo_{\alpha(\eta)}(M)\right) \widetilde{\subseteq} Clo_{\lambda}(f_{nu}(M))$.

Thus, we have $Clo_{\lambda}(f_{nu}(G)) = Clo_{\lambda}(f_{nu}(L))$ and $Clo_{\lambda}(f_{nu}(H)) = Clo_{\lambda}(f_{nu}(M))$. By Lemma 1, $Clo_{\eta}(G) = Clo_{\alpha(\eta)}(G)$ and $Clo_{\eta}(H) = Clo_{\alpha(\eta)}(H)$. So, we have $b_y \widetilde{\in} f_{nu}(Clo_{\eta}(G)) = f_{nu}\left(Clo_{\alpha(\eta)}(G)\right) \widetilde{\subseteq} Clo_{\lambda}(f_{nu}(L))$ and $d_z \widetilde{\in} f_{nu}(Clo_{\eta}(H)) = f_{nu}\left(Clo_{\alpha(\eta)}(H)\right) \widetilde{\subseteq} Clo_{\lambda}(f_{nu}(M))$. Let

$U = b_y \widetilde{\cup} f_{nu}(L)$ and $V = d_z \widetilde{\cup} f_{nu}(M)$.

Claim 1. $U, V \in SO(\lambda)$.

2. $f_{nu}(L) \widetilde{\cap} f_{nu}(M) = 0_{\mathcal{S}}$.

3. $U \widetilde{\cap} V = 0_{\mathcal{S}}$.

Proof of Claim. 1. Since f_{nu} is soft semi-open, $f_{nu}(L), f_{nu}(M) \in SO(\lambda)$. Since $f_{nu}(L) \widetilde{\subseteq} U \widetilde{\subseteq} Clo_\lambda(f_{nu}(L))$ and $f_{nu}(M) \widetilde{\subseteq} V \widetilde{\subseteq} Clo_\lambda(f_{nu}(M))$, then by Theorem 3.3 of [45], $U, V \in SO(\lambda)$.

2. Since $L, M \in \eta$ and $L \widetilde{\cap} M \widetilde{\subseteq} G \widetilde{\cap} H = 0_{\mathcal{T}}$, then by soft optimality of f_{nu}, $f_{nu}(L) \widetilde{\cap} f_{nu}(M) = 0_{\mathcal{S}}$.

3. Since $f_{nu} : (Z, \eta, \mathcal{T}) \longrightarrow (Y, \lambda, \mathcal{S})$ is soft closed-open, then

$f_{nu}(Clo_\eta(G)), f_{nu}(Clo_\eta(H)) \in \lambda$.

So, by Lemma 3 of [54], we have

$f_{nu}(Clo_\eta(G)) = Int_\lambda\left(f_{nu}\left(Clo_{\alpha(\eta)}(G)\right)\right) \widetilde{\subseteq} Int_\lambda\left(Clo_\lambda(f_{nu}(L)) \widetilde{\cup} f_{nu}(L) = sClo_\lambda(f_{nu}(L))\right.$

and

$f_{nu}(Clo_\eta(H)) = Int_\lambda\left(f_{nu}\left(Clo_{\alpha(\eta)}(H)\right)\right) \widetilde{\subseteq} Int_\lambda\left(Clo_\lambda(f_{nu}(M)) \widetilde{\cup} f_{nu}(M) = sClo_\lambda(f_{nu}(M))\right.$.

Therefore, we have

$$\begin{array}{cc}
U \widetilde{\cap} V & = \\
\left(b_y \widetilde{\cup} f_{nu}(L)\right) \widetilde{\cap} \left(d_z \widetilde{\cup} f_{nu}(M)\right) & = \\
\left(b_y \widetilde{\cap} f_{nu}(M)\right) \widetilde{\cup} \left(d_z \widetilde{\cap} f_{nu}(L)\right) \widetilde{\cup} \left(f_{nu}(L) \widetilde{\cap} f_{nu}(M)\right) & = \\
\left(b_y \widetilde{\cap} f_{nu}(M)\right) \widetilde{\cup} \left(d_z \widetilde{\cap} f_{nu}(L)\right) & \widetilde{\subseteq} \\
\left(sClo_\lambda(f_{nu}(L)) \widetilde{\cap} f_{nu}(M)\right) \widetilde{\cup} \left(sClo_\lambda(f_{nu}(M)) \widetilde{\cap} f_{nu}(L)\right) & = \\
0_{\mathcal{S}}. &
\end{array}$$

Therefore, we have $b_y \widetilde{\in} U \in SO(\lambda)$, $d_z \widetilde{\in} V \in SO(\lambda)$, and $U \widetilde{\cap} V = 0_{\mathcal{S}}$. Hence, $(Y, \lambda, \mathcal{S})$ is soft semi-Hausdorff.

The following two examples represent noteworthy comparisons between Theorems 4 and 5:

There is a soft pre-continuous map which is soft α-open but not soft α-continuous.

Example 6. Let $Z = \{1,2,3,4\}$, $Y = \{1,2,3\}$, $\mathcal{T} = \{a\}$, $\eta = \left\{0_{\mathcal{T}}, 1_T, a_{\{1,2\}}, a_{\{3,4\}}\right\}$, and $\lambda = \left\{0_{\mathcal{T}}, 1_{\mathcal{T}}, a_{\{2\}}, a_{\{1,2\}}, a_{\{2,3\}}\right\}$. Define the surjections $n : Z \longrightarrow Y$ and $u : \mathcal{T} \longrightarrow \mathcal{T}$ by $n(1) = 1$, $n(2) = n(3) = 2$, $n(4) = 3$, and $u(a) = a$. Then $f_{nu} : (Z, \eta, \mathcal{T}) \longrightarrow (Y, \lambda, \mathcal{T})$ is soft pre-continuous and soft α-open but not soft α-continuous.

There is a soft α-continuous which is soft semi-open but not soft α-open.

Example 7. Let $Z = \{1,2,3\}$, $\mathcal{T} = \{a\}$, $\eta = \left\{0_{\mathcal{T}}, 1_T, a_{\{1\}}, a_{\{2\}}, a_{\{1,2\}}, a_{\{1,3\}}\right\}$, and $\lambda = \left\{0_{\mathcal{T}}, 1_T, a_{\{1\}}, a_{\{2\}}, a_{\{1,2\}}\right\}$. Consider the identity maps $n : Z \longrightarrow X$ and $u : \mathcal{T} \longrightarrow \mathcal{T}$. Then $f_{nu} : (Z, \eta, \mathcal{T}) \longrightarrow (X, \lambda, \mathcal{T})$ is soft α-continuous and soft semi-open but not soft α-open.

Now we define soft strong α-continuity, which is a strong form of soft α-continuity:

Definition 6. A soft map $f_{nu} : (Z, \eta, \mathcal{T}) \longrightarrow (Y, \lambda, \mathcal{S})$ is called "soft strongly α-continuous" if $f_{nu}^{-1}(G) \in \alpha(\eta)$ for every $G \in SO(\lambda)$.

The following characterization of soft strong α-continuity will be used in Theorems 6 and 9:

Proposition 1. A soft map $f_{nu} : (Z, \eta, \mathcal{T}) \longrightarrow (Y, \lambda, \mathcal{S})$ is soft strongly α-continuous iff $f_{nu}\left(Clo_{\alpha(\eta)}(K)\right) \widetilde{\subseteq} sClo_\lambda(f_{nu}(K))$ for every $K \in SS(Z, \mathcal{T})$.

Proof. *Necessity.* Let $f_{nu} : (Z, \eta, \mathcal{T}) \longrightarrow (Y, \lambda, \mathcal{S})$ be soft strongly α-continuous. Let $K \in SS(Z, \mathcal{T})$. Then $1_{\mathcal{S}} - sClo_{\lambda}(f_{nu}(K)) \in SO(\lambda)$, and so,

$f_{nu}^{-1}(1_{\mathcal{S}} - sClo_{\lambda}(f_{nu}(K))) = 1_{\mathcal{T}} - f_{nu}^{-1}(sClo_{\lambda}(f_{nu}(K))) \in \alpha(\eta)$, and so, $f_{nu}^{-1}(sClo_{\lambda}(f_{nu}(K))) \in (\alpha(\eta))^{c}$.

Since $K \widetilde{\subseteq} f_{nu}^{-1}(f_{nu}(K)) \widetilde{\subseteq} f_{nu}^{-1}(sClo_{\lambda}(f_{nu}(K)) \in (\alpha(\eta))^{c}$, then

$Clo_{\alpha(\eta)}(K) \widetilde{\subseteq} f_{nu}^{-1}(sClo_{\lambda}(f_{nu}(K))$

and thus,

$f_{nu}\left(Clo_{\alpha(\eta)}(K)\right) \widetilde{\subseteq} f_{nu}\left(f_{nu}^{-1}(sClo_{\lambda}(f_{nu}(K))\right) \widetilde{\subseteq} sClo_{\lambda}(f_{nu}(K))$.

Sufficiency. Let $f_{nu}\left(Clo_{\alpha(\eta)}(K)\right) \widetilde{\subseteq} sClo_{\lambda}(f_{nu}(K))$ for every $K \in SS(Z, \mathcal{T})$. Let $G \in SO(\lambda)$. Then $1_{\mathcal{S}} - G \in SC(\lambda)$, and by assumption,

$f_{nu}\left(Clo_{\alpha(\eta)}\left(1_{\mathcal{T}} - f_{nu}^{-1}(G)\right)\right) \widetilde{\subseteq} sClo_{\lambda}(f_{nu}(1_{\mathcal{T}} - f_{nu}^{-1}(G)))$.

So,

$$\begin{array}{rcl} Clo_{\alpha(\eta)}\left(1_{\mathcal{T}} - f_{nu}^{-1}(G)\right) & \widetilde{\subseteq} & f_{nu}^{-1}\left(f_{nu}\left(Clo_{\alpha(\eta)}\left(1_{\mathcal{T}} - f_{nu}^{-1}(G)\right)\right)\right) \\ & \widetilde{\subseteq} & f_{nu}^{-1}\left(sClo_{\lambda}(f_{nu}(1_{\mathcal{T}} - f_{nu}^{-1}(G)))\right) \\ & = & f_{nu}^{-1}\left(sClo_{\lambda}(f_{nu}(f_{nu}^{-1}(1_{\mathcal{S}} - G)))\right) \\ & \widetilde{\subseteq} & f_{nu}^{-1}(sClo_{\lambda}(1_{\mathcal{S}} - G))) \\ & = & f_{nu}^{-1}(1_{\mathcal{S}} - G) \\ & & 1_{\mathcal{T}} - f_{nu}^{-1}(G). \end{array}$$

Therefore, $1_{\mathcal{T}} - f_{nu}^{-1}(G) \in (\alpha(\eta))^{c}$ and hence $f_{nu}^{-1}(G) \in \alpha(\eta)$.

Soft closed-openness can be eliminated if soft strong α-continuity is substituted for soft α-continuity in Theorem 5's assumptions.

Theorem 6. Let $f_{nu} : (Z, \eta, \mathcal{T}) \longrightarrow (Y, \lambda, \mathcal{S})$ be soft surjective, soft strongly α-continuous, soft semi-open, and soft optimal. If $(Z, \eta, \mathcal{T})$ is soft semi-Hausdorff, then $(Y, \lambda, \mathcal{S})$ is soft semi-Hausdorff.

Proof. Let $b_y, d_z \in SP(Y, \mathcal{S})$ such that $b_y \neq d_z$. Since f_{nu} is soft surjective, then we find $a_x, e_w \in SP(Z, \mathcal{T})$ such that $f_{nu}(a_x) = b_y$ and $f_{nu}(e_w) = d_z$. Since $b_y \neq d_z$, then $a_x \neq e_w$. Since $(Z, \eta, \mathcal{T})$ is soft semi-Hausdorff, we find $G, H \in SO(\eta)$ such that $a_x \widetilde{\in} G$, $e_w \widetilde{\in} H$, and $G \widetilde{\cap} H = 0_{\mathcal{T}}$. As $G, H \in SO(\eta)$, we find $L, M \in \eta$ such that $L \widetilde{\subseteq} G \widetilde{\subseteq} Clo_{\eta}(L)$ and $M \widetilde{\subseteq} H \widetilde{\subseteq} Clo_{\eta}(M)$. By Lemma 1, $Clo_{\eta}(L) = Clo_{\alpha(\eta)}(L)$ and $Clo_{\eta}(M) = Clo_{\alpha(\eta)}(M)$. Since $f_{nu} : (Z, \eta, \mathcal{T}) \longrightarrow (Y, \lambda, \mathcal{S})$ is soft strongly α-continuous, then by Proposition 1,

$f_{nu}\left(Clo_{\alpha(\eta)}(L)\right) \widetilde{\subseteq} sClo_{\lambda}(f_{nu}(L))$ and $f_{nu}\left(Clo_{\alpha(\eta)}(M)\right) \widetilde{\subseteq} sClo_{\lambda}(f_{nu}(M))$.

So, we have

$f_{nu}(L) \widetilde{\subseteq} f_{nu}(G) \widetilde{\subseteq} f_{nu}(Clo_{\eta}(L)) = f_{nu}\left(Clo_{\alpha(\eta)}(L)\right) \widetilde{\subseteq} sClo_{\lambda}(f_{nu}(L))$

and

$f_{nu}(M) \widetilde{\subseteq} f_{nu}(H) \widetilde{\subseteq} f_{nu}(Clo_{\eta}(M)) = f_{nu}\left(Clo_{\alpha(\eta)}(M)\right) \widetilde{\subseteq} sClo_{\lambda}(f_{nu}(M))$.

Hence,

$sClo_{\lambda}(f_{nu}(G)) = sClo_{\lambda}(f_{nu}(L))$ and $sClo_{\lambda}(f_{nu}(H)) = sClo_{\lambda}(f_{nu}(M))$.

By Lemma 1, $Clo_{\eta}(G) = Clo_{\alpha(\eta)}(G)$ and $Clo_{\eta}(H) = Clo_{\alpha(\eta)}(H)$.

Therefore, we have

$b_y \widetilde{\in} f_{nu}(Clo_{\eta}(G)) = f_{nu}\left(Clo_{\alpha(\eta)}(G)\right) \widetilde{\subseteq} sClo_{\lambda}(f_{nu}(L))$

and

$d_z \widetilde{\in} f_{nu}(Clo_{\eta}(H)) = f_{nu}\left(Clo_{\alpha(\eta)}(H)\right) \widetilde{\subseteq} sClo_{\lambda}(f_{nu}(M))$.

Let $U = b_y \widetilde{\cup} f_{nu}(L)$ and $V = d_z \widetilde{\cup} f_{nu}(M)$.

Claim 1. $U, V \in SO(\lambda)$.

2. $f_{nu}(L) \widetilde{\cap} f_{nu}(M) = 0_{\mathcal{S}}$.

3. $U \widetilde{\cap} V = 0_{\mathcal{S}}$.

Proof of Claim. 1. Since f_{nu} is soft semi-open, $f_{nu}(L), f_{nu}(M) \in SO(\eta)$. Since $f_{nu}(L) \widetilde{\subseteq} U \widetilde{\subseteq} Clo_\lambda(f_{nu}(L))$ and $f_{nu}(M) \widetilde{\subseteq} V \widetilde{\subseteq} Clo_\lambda(f_{nu}(M))$, then by Theorem 3.3 of [45], $U, V \in SO(\lambda)$.

2. Since $L, M \in \eta$ and $L \widetilde{\cap} M \widetilde{\subseteq} G \widetilde{\cap} H = 0_{\mathcal{T}}$, then by soft optimality of f_{nu}, $f_{nu}(L) \widetilde{\cap} f_{nu}(M) = 0_{\mathcal{S}}$.

3.
$$\begin{array}{cc}
U \widetilde{\cap} V & = \\
\left(b_y \widetilde{\cup} f_{nu}(L)\right) \widetilde{\cap} \left(d_z \widetilde{\cup} f_{nu}(M)\right) & = \\
\left(b_y \widetilde{\cap} f_{nu}(M)\right) \widetilde{\cup} \left(d_z \widetilde{\cap} f_{nu}(L)\right) \widetilde{\cup} \left(f_{nu}(L) \widetilde{\cap} f_{nu}(M)\right) & = \\
\left(b_y \widetilde{\cap} f_{nu}(M)\right) \widetilde{\cup} \left(d_z \widetilde{\cap} f_{nu}(L)\right) & \widetilde{\subseteq} \\
\left(sClo_\lambda(f_{nu}(L)) \widetilde{\cap} f_{nu}(M)\right) \widetilde{\cup} \left(sClo_\lambda(f_{nu}(M)) \widetilde{\cap} f_{nu}(L)\right) & = \\
0_{\mathcal{S}}. &
\end{array}$$

Therefore, we have $b_y \widetilde{\in} U \in SO(\lambda)$, $d_z \widetilde{\in} V \in SO(\lambda)$, and $U \widetilde{\cap} V = 0_{\mathcal{S}}$. Hence, $(Y, \lambda, \mathcal{S})$ is soft semi-Hausdorff.

The following characterization of soft semi-continuity will be used in Theorem 7 and Proposition 3:

Proposition 2. A soft map $f_{nu} : (Z, \eta, \mathcal{T}) \longrightarrow (Y, \lambda, \mathcal{S})$ is soft semi-continuous iff $f_{nu}(Int_\eta(Clo_\eta(K))) \widetilde{\subseteq} Clo_\lambda(f_{nu}(K))$ for every $K \in SS(Z, \mathcal{T})$.

Proof. *Necessity.* Let $f_{nu} : (Z, \eta, \mathcal{T}) \longrightarrow (Y, \lambda, \mathcal{S})$ be soft semi-continuous. Let $K \in SS(Z, \mathcal{T})$. Then $1_{\mathcal{S}} - Clo_\lambda(f_{nu}(K)) \in \lambda$, and so,

$f_{nu}^{-1}(1_{\mathcal{S}} - Clo_\lambda(f_{nu}(K))) = 1_{\mathcal{T}} - f_{nu}^{-1}(Clo_\lambda(f_{nu}(K))) \in SO(\eta)$, and so, $f_{nu}^{-1}(Clo_\lambda(f_{nu}(K))) \in SC(\eta)$.

Since $K \widetilde{\subseteq} f_{nu}^{-1}(f_{nu}(K)) \widetilde{\subseteq} f_{nu}^{-1}(Clo_\lambda(f_{nu}(K)) \in SC(\eta)$, then

$sClo_\eta(K) \widetilde{\subseteq} f_{nu}^{-1}(Clo_\lambda(f_{nu}(K))$. So, by Lemma 3 of [54],

$Int_\eta(Clo_\eta(K)) \widetilde{\subseteq} f_{nu}^{-1}(Clo_\lambda(f_{nu}(K))$, and thus,

$f_{nu}(Int_\eta(Clo_\eta(K))) \widetilde{\subseteq} f_{nu}\left(f_{nu}^{-1}(Clo_\lambda(f_{nu}(K))\right) \widetilde{\subseteq} Clo_\lambda(f_{nu}(K))$.

Sufficiency. Let $f_{nu}(Int_\eta(Clo_\eta(K))) \widetilde{\subseteq} Clo_\lambda(f_{nu}(K))$ for every $K \in SS(Z, \mathcal{T})$. Let $G \in \lambda$. Then $1_{\mathcal{S}} - G \in \lambda^c$ and by assumption,

$f_{nu}\left(Int_\eta\left(Clo_\eta(1_{\mathcal{T}} - f_{nu}^{-1}(G))\right)\right) \widetilde{\subseteq} Clo_\lambda(f_{nu}(1_{\mathcal{T}} - f_{nu}^{-1}(G)))$.

So,

$$\begin{array}{ccc}
Int_\eta\left(Clo_\eta(1_{\mathcal{T}} - f_{nu}^{-1}(G))\right) & \widetilde{\subseteq} & f_{nu}^{-1}\left(f_{nu}\left(Int_\eta\left(Clo_\eta(1_{\mathcal{T}} - f_{nu}^{-1}(G))\right)\right)\right) \\
& \widetilde{\subseteq} & f_{nu}^{-1}\left(Clo_\lambda(f_{nu}(1_{\mathcal{T}} - f_{nu}^{-1}(G)))\right) \\
& = & f_{nu}^{-1}\left(Clo_\lambda(f_{nu}(f_{nu}^{-1}(1_{\mathcal{S}} - G)))\right) \\
& \widetilde{\subseteq} & f_{nu}^{-1}(Clo_\lambda(1_{\mathcal{S}} - G))) \\
& = & f_{nu}^{-1}(1_{\mathcal{S}} - G) \\
& & 1_{\mathcal{T}} - f_{nu}^{-1}(G).
\end{array}$$

Thus, $Int_\eta\left(Clo_\eta(1_{\mathcal{T}} - f_{nu}^{-1}(G))\right) \widetilde{\cup} \left(1_{\mathcal{T}} - f_{nu}^{-1}(G)\right) \widetilde{\subseteq} 1_{\mathcal{T}} - f_{nu}^{-1}(G)$ and by Lemma 3 of [54], $sClo_\eta(1_{\mathcal{T}} - f_{nu}^{-1}(G)) \widetilde{\subseteq} 1_{\mathcal{T}} - f_{nu}^{-1}(G)$. It follows that $1_{\mathcal{T}} - f_{nu}^{-1}(G) \in SC(\eta)$; hence $f_{nu}^{-1}(G) \in SO(\eta)$.

The next result is a new preservation theorem for soft Hausdorff spaces:

Theorem 7. Let $f_{nu} : (Z, \eta, \mathcal{T}) \longrightarrow (Y, \lambda, \mathcal{S})$ be soft surjective, soft semi-continuous, soft pre-semi-closed, soft pre-open, and soft optimal. If $(Z, \eta, \mathcal{T})$ is soft Hausdorff, then $(Y, \lambda, \mathcal{S})$ is soft Hausdorff.

Proof. Let $b_y, d_z \in SP(Y, \mathcal{S})$ such that $b_y \neq d_z$. Since f_{nu} is soft surjective, then we find $a_x, e_w \in SP(Z, \mathcal{T})$ such that $f_{nu}(a_x) = b_y$ and $f_{nu}(e_w) = d_z$. Since $b_y \neq d_z$, then $a_x \neq e_w$. Since $(Z, \eta, \mathcal{T})$ is soft Hausdorff, we find $G, H \in \eta \subseteq \alpha(\eta)$ such that $a_x \widetilde{\in} G$, $e_w \widetilde{\in} H$, and $G \widetilde{\cap} H = 0_{\mathcal{T}}$. As $G, H \in \alpha(\eta)$, we find $L, M \in \eta$ such that $L \widetilde{\subseteq} G \widetilde{\subseteq} Int_\eta(Clo_\eta(L))$ and $M \widetilde{\subseteq} H \widetilde{\subseteq} Int_\eta(Clo_\eta(M))$. Since f_{nu} is soft semi-continuous, then by Proposition 2,

$$f_{nu}(Int_\eta(Clo_\eta(L))) \widetilde{\subseteq} Clo_\lambda(f_{nu}(L))$$

and

$$f_{nu}(Int_\eta(Clo_\eta(M))) \widetilde{\subseteq} Clo_\lambda(f_{nu}(M)).$$

Since f_{nu} is soft pre-open, then

$$\begin{array}{rcc} f_{nu}(G) & \widetilde{\subseteq} & f_{nu}(Int_\eta(Clo_\eta(L))) \\ & \widetilde{\subseteq} & Int_\lambda(Clo_\lambda(f_{nu}(Int_\eta(Clo_\eta(L))))) \\ & \widetilde{\subseteq} & Int_\lambda(Clo_\lambda(f_{nu}(L))) \end{array}$$

and

$$\begin{array}{rcc} f_{nu}(H) & \widetilde{\subseteq} & f_{nu}(Int_\eta(Clo_\eta(M))) \\ & \widetilde{\subseteq} & Int_\lambda(Clo_\lambda(f_{nu}(Int_\eta(Clo_\eta(M))))) \\ & \widetilde{\subseteq} & Int_\lambda(Clo_\lambda(f_{nu}(M))). \end{array}$$

So, we have the following two inclusions:

$$f_{nu}(G) \widetilde{\subseteq} Int_\lambda(Clo_\lambda(f_{nu}(L)))$$

and

$$f_{nu}(H) \widetilde{\subseteq} Int_\lambda(Clo_\lambda(f_{nu}(M))) \widetilde{\subseteq} Int_\lambda(Clo_\lambda(f_{nu}(Int_\eta(Clo_\eta(M))))).$$

Thus, we have

$$b_y \widetilde{\in} Int_\lambda(Clo_\lambda(f_{nu}(L))) \in \lambda \text{ and } d_z \widetilde{\in} Int_\lambda(Clo_\lambda(f_{nu}(Int_\eta(Clo_\eta(M))))) \in \lambda.$$

Since $L \widetilde{\cap} M \widetilde{\subseteq} G \widetilde{\cap} H = 0_{\mathcal{T}}$, then $L \widetilde{\cap} (Int_\eta(Clo_\eta(M))) = 0_{\mathcal{T}}$.

Since f_{nu} is soft optimal, then $f_{nu}(L) \widetilde{\cap} f_{nu}(Int_\eta(Clo_\eta(M))) = 0_{\mathcal{S}}$.

Since f_{nu} is soft pre-semi-closed, then

$$f_{nu}(L) \widetilde{\cap} Int_\lambda(Clo_\lambda(f_{nu}(Int_\eta(Clo_\eta(M))))) = 0_{\mathcal{S}}.$$

So,

$$(Int_\lambda(Clo_\lambda(f_{nu}(L)))) \widetilde{\cap} Int_\lambda(Clo_\lambda(f_{nu}(Int_\eta(Clo_\eta(M))))) =$$

$$Int_\lambda\left(Clo_\lambda\left(f_{nu}(L) \widetilde{\cap} f_{nu}(Int_\eta(Clo_\eta(M)))\right)\right) = 0_{\mathcal{S}}.$$

This shows that $(Y, \lambda, \mathcal{S})$ is soft Hausdorff.

The proofs of Theorems 8 and 9 as well as Proposition 3 will make use of the following lemma:

Lemma 3. Let $(Z, \eta, \mathcal{T})$ be a STS and let $K \in SS(X, \mathcal{T})$. Then $K \in RO(\eta)$ iff $K \in \eta \cap SC(\eta)$.

Proof. *Necessity.* Let $K \in RO(\eta)$. Then $K = Int_\eta(Clo_\eta(K))$. So, $K = Int_\eta(Clo_\eta(K)) \in \eta$. Since $Int_\eta(Clo_\eta(K)) \widetilde{\subseteq} K$, then $K \in \eta \cap SC(\eta)$.

Sufficiency. Since $K \in \eta$, then $K \widetilde{\subseteq} Int_\eta(Clo_\eta(K))$. On the other hand, since $K \in SC(\eta)$, then $Int_\eta(Clo_\eta(K)) \widetilde{\subseteq} K$. Thus, $K = Int_\eta(Clo_\eta(K))$ and hence $K \in RO(\eta)$.

Now we define soft RO-pre-semi-closed maps, which are a subclass of soft pre-semi-closed maps:

Definition 7. A soft map $f_{nu}:(Z,\eta,\mathcal{T})\longrightarrow(Y,\lambda,\mathcal{S})$ is called "soft RO-pre-semi-closed" if $f_{nu}(F)\in RO(\lambda)$ for every $F\in SC(\eta)$.

Theorem 8. Every soft RO-pre-semi-closed map is soft pre-semi-closed.

Proof. Follows from Definitions 7, 4 (d) and Lemma 3.

In general, the opposite of Theorem 8 need not be true:

Example 8. Let $Z=\{1,2,3\}$, $\mathcal{T}=\mathbb{R}$, and $\eta=\left\{0_{\mathcal{T}},1_{T},C_{\{1\}}\right\}$. Consider the identity maps $n:Z\longrightarrow Z$ and $u:\mathcal{T}\longrightarrow\mathcal{T}$. Then $f_{nu}:(Z,\eta,\mathcal{T})\longrightarrow(Y,\lambda,\mathcal{T})$ is soft pre-semi-closed but not soft RO-pre-semi-closed.

The following result says that the soft semi-continuous map that is soft RO-pre-semi-closed is soft pre-open:

Proposition 3. If $f_{nu}:(Z,\eta,\mathcal{T})\longrightarrow(Y,\lambda,\mathcal{S})$ is soft semi-continuous and soft RO-pre-semi-closed, then f_{nu} is soft pre-open.

Proof. Let $G\in\eta$. Since f_{nu} is soft semi-continuous, by Proposition 2, $f_{nu}(G)\widetilde{\subseteq}f_{nu}(Int_{\eta}(Clo_{\eta}(G)))\widetilde{\subseteq}Clo_{\lambda}(f_{nu}(G))$. So,

$Clo_{\lambda}(f_{nu}(G))\widetilde{\subseteq}Clo_{\lambda}(f_{nu}(Int_{\eta}(Clo_{\eta}(G))))\widetilde{\subseteq}Clo_{\lambda}(f_{nu}(G))$,

and hence

$Clo_{\lambda}(f_{nu}(G))=Clo_{\lambda}(f_{nu}(Int_{\eta}(Clo_{\eta}(G))))$.

Thus,

$Int_{\lambda}(Clo_{\lambda}(f_{nu}(G)))=Int_{\lambda}(Clo_{\lambda}(f_{nu}(Int_{\eta}(Clo_{\eta}(G)))))$.

Since $Int_{\eta}(Clo_{\eta}(G))\in RO(\eta)$, then by Lemma 3, $Int_{\eta}(Clo_{\eta}(G))\in SC(\eta)$. Since f_{nu} is soft RO-pre-semi-closed, then

$Int_{\lambda}(Clo_{\lambda}(f_{nu}(Int_{\eta}(Clo_{\eta}(G)))))=f_{nu}(Int_{\eta}(Clo_{\eta}(G)))$.

So, we have

$f_{nu}(G)\widetilde{\subseteq}f_{nu}(Int_{\eta}(Clo_{\eta}(G)))=Int_{\lambda}(Clo_{\lambda}(f_{nu}(G)))$.

Hence, $f_{nu}(G)\in PO(\lambda)$. This shows that f_{nu} is soft pre-open.

In the following result, we provide sufficient conditions for a soft optimal map for sending a soft semi-Hausdorff space onto a soft Hausdorff space.

Theorem 9. Let $f_{nu}:(Z,\eta,\mathcal{T})\longrightarrow(Y,\lambda,\mathcal{S})$ be soft surjective, soft strongly α-continuous, soft pre-semi-closed, soft pre-open, and soft optimal. If $(Z,\eta,\mathcal{T})$ is soft semi-Hausdorff, then $(Y,\lambda,\mathcal{S})$ is soft Hausdorff.

Proof. Let $b_y,d_z\in SP(Y,\mathcal{S})$ such that $b_y\neq d_z$. Since f_{nu} is soft surjective, then we find $a_x,e_w\in SP(Z,\mathcal{T})$ such that $f_{nu}(a_x)=b_y$ and $f_{nu}(e_w)=d_z$. Since $b_y\neq d_z$, then $a_x\neq e_w$. Since $(Z,\eta,\mathcal{T})$ is soft semi-Hausdorff, we find $G,H\in SO(\eta)$ such that $a_x\widetilde{\in}G$, $e_w\widetilde{\in}H$, and $G\widetilde{\cap}H=0_{\mathcal{T}}$. As $G,H\in SO(\eta)$, we find $L,M\in\eta$ such that $L\widetilde{\subseteq}G\widetilde{\subseteq}Clo_{\eta}(L)$ and $M\widetilde{\subseteq}H\widetilde{\subseteq}Clo_{\eta}(M)$. By Lemma 1, $Clo_{\eta}(L)=Clo_{\alpha(\eta)}(L)$ and $Clo_{\eta}(M)=Clo_{\alpha(\eta)}(M)$. Since $f_{nu}:(Z,\eta,\mathcal{T})\longrightarrow(Y,\lambda,\mathcal{S})$ is soft strongly α-continuous, then by Proposition 1,

$f_{nu}\left(Clo_{\alpha(\eta)}(L)\right)\widetilde{\subseteq}sClo_{\lambda}(f_{nu}(L))$ and $f_{nu}\left(Clo_{\alpha(\eta)}(M)\right)\widetilde{\subseteq}sClo_{\lambda}(f_{nu}(M))$.

So, we have

$f_{nu}(L)\widetilde{\subseteq}f_{nu}(G)\widetilde{\subseteq}f_{nu}(Clo_{\eta}(L))=f_{nu}\left(Clo_{\alpha(\eta)}(L)\right)\widetilde{\subseteq}sClo_{\lambda}(f_{nu}(L))$

and

$f_{nu}(M)\widetilde{\subseteq}f_{nu}(H)\widetilde{\subseteq}f_{nu}(Clo_\eta(M))=f_{nu}\left(Clo_{\alpha(\eta)}(M)\right)\widetilde{\subseteq}sClo_\lambda(f_{nu}(M))$.

Thus, we have

$sClo_\lambda(f_{nu}(G))=sClo_\lambda(f_{nu}(L))$ and $sClo_\lambda(f_{nu}(H))=sClo_\lambda(f_{nu}(M))$.

Since f_{nu} is soft pre-open, $f_{nu}(L), f_{nu}(M)\in PO(\lambda)$; hence

$f_{nu}(L)\widetilde{\subseteq}Int_\lambda(Clo_\lambda(f_{nu}(L)))$ and $f_{nu}(M)\widetilde{\subseteq}Int_\lambda(Clo_\lambda(f_{nu}(M)))$.

So, by Lemma 3 of [54],

$$\begin{array}{rcl} sClo_\lambda(f_{nu}(G)) & = & sClo_\lambda(f_{nu}(L)) \\ & = & Int_\lambda(Clo_\lambda(f_{nu}(L)))\widetilde{\cup}f_{nu}(L) \\ & = & Int_\lambda(Clo_\lambda(f_{nu}(L))) \end{array}$$

and

$$\begin{array}{rcl} sClo_\lambda(f_{nu}(H)) & = & sClo_\lambda(f_{nu}(M)) \\ & = & Int_\lambda(Clo_\lambda(f_{nu}(M)))\widetilde{\cup}f_{nu}(M) \\ & = & Int_\lambda(Clo_\lambda(f_{nu}(M))). \end{array}$$

Since $Int_\eta(Clo_\eta(G)), Int_\eta(Clo_\eta(H))\in RO(\eta)$, then by Lemma 3, $Int_\eta(Clo_\eta(L)), Int_\eta(Clo_\eta(M))\in SC(\eta)$. Since f_{nu} is soft pre-semi-closed, then $b_y\widetilde{\in}f_{nu}(G)\widetilde{\subseteq}Int_\lambda(Clo_\lambda(f_{nu}(Int_\eta(Clo_\eta(L)))))\widetilde{\subseteq}f_{nu}(Int_\eta(Clo_\eta(L)))$

and

$d_z\widetilde{\in}f_{nu}(H)\widetilde{\subseteq}Int_\lambda(Clo_\lambda(f_{nu}(Int_\eta(Clo_\eta(M)))))\widetilde{\subseteq}f_{nu}(Int_\eta(Clo_\eta(M)))$.

Since $G\widetilde{\cap}H=0_{\mathcal{T}}$, then $L\widetilde{\cap}M=0_{\mathcal{T}}$ and so, $Int_\eta(Clo_\eta(L))\widetilde{\cap}Int_\eta(Clo_\eta(M))=0_{\mathcal{T}}$. So, by soft optimality of f_{nu}, $f_{nu}(Int_\eta(Clo_\eta(L)))\widetilde{\cap}f_{nu}(Int_\eta(Clo_\eta(M)))=0_{\mathcal{S}}$; hence

$Int_\lambda(Clo_\lambda(f_{nu}(Int_\eta(Clo_\eta(L)))))\widetilde{\cap}Int_\lambda(Clo_\lambda(f_{nu}(Int_\eta(Clo_\eta(M)))))=0_{\mathcal{S}}$.

Therefore, we have

$b_y\widetilde{\in}Int_\lambda(Clo_\lambda(f_{nu}(Int_\eta(Clo_\eta(L)))))\in\lambda$, $d_z\widetilde{\in}Int_\lambda(Clo_\lambda(f_{nu}(Int_\eta(Clo_\eta(M)))))\in\lambda$, and

$Int_\lambda(Clo_\lambda(f_{nu}(Int_\eta(Clo_\eta(L)))))\widetilde{\cap}Int_\lambda(Clo_\lambda(f_{nu}(Int_\eta(Clo_\eta(M)))))=0_{\mathcal{S}}$.

Hence, $(Y,\lambda,\mathcal{S})$ is soft Hausdorff.

The following two lemmas will be used in the proof of Proposition 4.

Lemma 4. Let $(Z,\eta,\mathcal{T})$ be a STS. If $K\in SC(\eta)$ or $N\in SC(\eta)$, then $Int_\eta\left(Clo_\eta\left(K\widetilde{\cap}N\right)\right)=Int_\eta(Clo_\eta(K))\widetilde{\cap}Int_\eta(Clo_\eta(N))$.

Proof. We need only to show that $Int_\eta(Clo_\eta(K))\widetilde{\cap}Int_\eta(Clo_\eta(N))\widetilde{\subseteq}Int_\eta\left(Clo_\eta\left(K\widetilde{\cap}N\right)\right)$. Suppose that $K\in SC(\eta)$. Then $Int_\eta(Clo_\eta(K))\widetilde{\subseteq}K$. Since $Int_\eta(Clo_\eta(K)), Int_\eta(Clo_\eta(N))\in RO(\eta)$, then $Int_\eta(Clo_\eta(K))\widetilde{\cap}Int_\eta(Clo_\eta(N))\in RO(\eta)$. So,

$$\begin{array}{rcl} Int_\eta(Clo_\eta(K))\widetilde{\cap}Int_\eta(Clo_\eta(N)) & = & Int_\eta\left(Clo_\eta\left(Int_\eta(Clo_\eta(K))\widetilde{\cap}Int_\eta(Clo_\eta(N))\right)\right) \\ & \widetilde{\subseteq} & Int_\eta\left(Clo_\eta\left(Int_\eta(Clo_\eta(K))\widetilde{\cap}Clo_\eta(N)\right)\right) \\ & \widetilde{\subseteq} & Int_\eta\left(Clo_\eta\left(Int_\eta(Clo_\eta(K))\widetilde{\cap}N\right)\right) \\ & \widetilde{\subseteq} & Int_\eta\left(Clo_\eta\left(K\widetilde{\cap}N\right)\right). \end{array}$$

Lemma 5. Let $(Z,\eta,\mathcal{T})$ be a STS. If $K\in SO(\eta)$ or $N\in SO(\eta)$ and $K\widetilde{\cap}N=0_{\mathcal{T}}$, then

$Int_\eta(Clo_\eta(K))\widetilde{\cap}Int_\eta(Clo_\eta(N))=0_{\mathcal{T}}$.

Proof. Let $K\in SO(\eta)$. Then $K\widetilde{\subseteq}Clo_\eta(Int_\eta(K))$. So,

$$\begin{array}{rcl} Int_\eta\left(Clo_\eta(K)\right)\widetilde{\cap}Int_\eta\left(Clo_\eta(N)\right) & \widetilde{\subseteq} & Int_\eta\left(Clo_\eta(Clo_\eta\left(Int_\eta\left(K\right)\right))\right)\widetilde{\cap}Int_\eta\left(Clo_\eta(N)\right) \\ & = & Int_\eta\left(Clo_\eta\left(Int_\eta\left(K\right)\right)\right)\widetilde{\cap}Int_\eta\left(Clo_\eta(N)\right) \\ & \widetilde{\subseteq} & Clo_\eta\left(Int_\eta\left(K\right)\right)\widetilde{\cap}Int_\eta\left(Clo_\eta(N)\right). \end{array}$$

Claim. $Clo_\eta\left(Int_\eta\left(K\right)\right)\widetilde{\cap}Int_\eta\left(Clo_\eta(N)\right) = 0_{\mathcal{T}}$.

Proof of Claim. Suppose to the contrary that we find

$a_x\widetilde{\in}Clo_\eta\left(Int_\eta\left(K\right)\right)\widetilde{\cap}Int_\eta\left(Clo_\eta(N)\right)$.

Since $a_x\widetilde{\in}\ Int_\eta\left(Clo_\eta(N)\right) \in \eta$ and $a_x\widetilde{\in}Clo_\eta\left(Int_\eta\left(K\right)\right)$, then

$Int_\eta\left(Clo_\eta(N)\right)\widetilde{\cap}Int_\eta\left(K\right) \neq 0_{\mathcal{T}}$. Choose $d_z\widetilde{\in}Int_\eta\left(Clo_\eta(N)\right)\widetilde{\cap}Int_\eta\left(K\right)$,

then we have $d_z\widetilde{\in}Int_\eta\left(K\right) \in \eta$ and $d_z\widetilde{\in}Clo_\eta(N)$; hence, $N\widetilde{\cap}Int_\eta\left(K\right) \neq 0_{\mathcal{T}}$. On the other hand, since $N\widetilde{\cap}Int_\eta\left(K\right)\widetilde{\subseteq}K\widetilde{\cap}N = 0_{\mathcal{T}}$, then $N\widetilde{\cap}Int_\eta\left(K\right) = 0_{\mathcal{T}}$ which is a contradiction.

It follows that $Int_\eta\left(Clo_\eta(K)\right)\widetilde{\cap}Int_\eta\left(Clo_\eta(N)\right) = 0_{\mathcal{T}}$.

The following three propositions are essential to the proof of the main theorem that follows them.

Proposition 4. Let $f_{nu} : (Z, \eta, \mathcal{T}) \longrightarrow (Y, \lambda, \mathcal{S})$ be soft optimal. If $K \in SO\left(\eta\right) \cup SC\left(\eta\right)$ or $N \in SO\left(\eta\right) \cup SC\left(\eta\right)$, and $K\widetilde{\cap}N = 0_{\mathcal{T}}$, then

$f_{nu}\left(Int_\eta\left(Clo_\eta(K)\right)\right)\widetilde{\cap}f_{nu}\left(Int_\eta\left(Clo_\eta(N)\right)\right) = 0_{\mathcal{S}}$.

Proof. Suppose that $K \in SC\left(\eta\right)$. Then by Lemma 4,

$$\begin{array}{rcl} Int_\eta\left(Clo_\eta(K)\right)\widetilde{\cap}Int_\eta\left(Clo_\eta(N)\right) & = & Int_\eta\left(Clo_\eta\left(K\widetilde{\cap}N\right)\right) \\ & = & Int_\eta\left(Clo_\eta\left(0_{\mathcal{T}}\right)\right) \\ & = & 0_{\mathcal{T}}. \end{array}$$

Since $Int_\eta\left(Clo_\eta(K)\right), Int_\eta\left(Clo_\eta(N)\right) \in \eta$,

$Int_\eta\left(Clo_\eta(K)\right)\widetilde{\cap}Int_\eta\left(Clo_\eta(N)\right) = 0_{\mathcal{T}}$, and f_{nu} is soft optimal, then

$f_{nu}\left(Int_\eta\left(Clo_\eta(K)\right)\right)\widetilde{\cap}f_{nu}\left(Int_\eta\left(Clo_\eta(N)\right)\right) = 0_{\mathcal{S}}$.

Suppose that $K \in SO\left(\eta\right)$. Then by Lemma 5, $Int_\eta\left(Clo_\eta(K)\right)\widetilde{\cap}Int_\eta\left(Clo_\eta(N)\right) = 0_{\mathcal{T}}$. Since $Int_\eta\left(Clo_\eta(K)\right), Int_\eta\left(Clo_\eta(N)\right) \in \eta$, $Int_\eta\left(Clo_\eta(K)\right)\widetilde{\cap}Int_\eta\left(Clo_\eta(N)\right) = 0_{\mathcal{T}}$, and f_{nu} is soft optimal, then $f_{nu}\left(Int_\eta\left(Clo_\eta(K)\right)\right)\widetilde{\cap}f_{nu}\left(Int_\eta\left(Clo_\eta(N)\right)\right) = 0_{\mathcal{S}}$.

Proposition 5. Let $f_{nu} : (Z, \eta, \mathcal{T}) \longrightarrow (Y, \lambda, \mathcal{S})$ be soft optimal, soft almost-open, and soft pre-continuous. If $K \in SO\left(\eta\right) \cup SC\left(\eta\right)$ or $N \in SO\left(\eta\right) \cup SC\left(\eta\right)$, and $K\widetilde{\cap}N = 0_{\mathcal{T}}$, then

$f_{nu}\left(Int_\eta\left(Clo_\eta(K)\right)\right)\widetilde{\cap}f_{nu}\left(Clo_\eta\left(Int_\eta\left(Clo_\eta(N)\right)\right)\right) = 0_{\mathcal{S}}$.

Proof. As f_{nu} is soft optimal, then by Proposition 4,

$f_{nu}\left(Int_\eta\left(Clo_\eta(K)\right)\right)\widetilde{\cap}f_{nu}\left(Int_\eta\left(Clo_\eta(N)\right)\right) = 0_{\mathcal{S}}$.

As f_{nu} is soft almost-open, then $f_{nu}\left(Int_\eta\left(Clo_\eta(K)\right)\right) \in \lambda$.

Hence, $f_{nu}\left(Int_\eta\left(Clo_\eta(K)\right)\right)\widetilde{\cap}Clo_\lambda\left(f_{nu}\left(Int_\eta\left(Clo_\eta(N)\right)\right)\right) = 0_{\mathcal{S}}$.

As f_{nu} is soft pre-continuous, then by Theorem 18 of [53],

$f_{nu}\left(Clo_\eta\left(Int_\eta\left(Clo_\eta(N)\right)\right)\right)\widetilde{\subseteq}Clo_\lambda\left(f_{nu}\left(Int_\eta\left(Clo_\eta(N)\right)\right)\right)$.

Thus,

$$\begin{array}{rl} f_{nu}\left(Int_{\eta}\left(Clo_{\eta}(K)\right)\right)\widetilde{\cap}f_{nu}\left(Clo_{\eta}\left(Int_{\eta}\left(Clo_{\eta}(N)\right)\right)\right) & \widetilde{\subseteq} \\ f_{nu}\left(Int_{\eta}\left(Clo_{\eta}(K)\right)\right)\widetilde{\cap}Clo_{\lambda}\left(f_{nu}\left(Int_{\eta}\left(Clo_{\eta}(N)\right)\right)\right) & = \\ & 0_{\mathcal{S}}. \end{array}$$

This ends the proof.

Proposition 6. Let $f_{nu}:(Z,\eta,\mathcal{T})\longrightarrow(Y,\lambda,\mathcal{S})$ be soft optimal, soft s-open, and soft pre-continuous. If $K\in SO(\eta)\cup SC(\eta)$ or $N\in SO(\eta)\cup SC(\eta)$, and $K\widetilde{\cap}N=0_{\mathcal{T}}$, then

$f_{nu}\left(Clo_{\eta}\left(Int_{\eta}\left(Clo_{\eta}(K)\right)\right)\right)\widetilde{\cap}f_{nu}\left(Clo_{\eta}\left(Int_{\eta}\left(Clo_{\eta}(N)\right)\right)\right)=0_{\mathcal{S}}$.

Proof. Since every soft s-open map is soft almost-open, then by Proposition 5,

$f_{nu}\left(Int_{\eta}\left(Clo_{\eta}(K)\right)\right)\widetilde{\cap}f_{nu}\left(Clo_{\eta}\left(Int_{\eta}\left(Clo_{\eta}(N)\right)\right)\right)=0_{\mathcal{S}}$. Since f_{nu} is soft s-open and $Clo_{\eta}\left(Int_{\eta}\left(Clo_{\eta}(N)\right)\right)\in SC(\eta)$, then

$f_{nu}\left(Clo_{\eta}\left(Int_{\eta}\left(Clo_{\eta}(N)\right)\right)\right)\in\lambda$. Hence,

$Clo_{\lambda}\left(f_{nu}\left(Int_{\eta}\left(Clo_{\eta}(K)\right)\right)\right)\widetilde{\cap}f_{nu}\left(Clo_{\eta}\left(Int_{\eta}\left(Clo_{\eta}(N)\right)\right)\right)=0_{\mathcal{S}}$.

As f_{nu} is soft pre-continuous, then by Theorem 18 of [53],

$f_{nu}\left(Clo_{\eta}\left(Int_{\eta}\left(Clo_{\eta}(K)\right)\right)\right)\widetilde{\subseteq}Clo_{\lambda}\left(f_{nu}\left(Int_{\eta}\left(Clo_{\eta}(K)\right)\right)\right)$. Thus,

$$\begin{array}{rl} f_{nu}\left(Clo_{\eta}\left(Int_{\eta}\left(Clo_{\eta}(K)\right)\right)\right)\widetilde{\cap}f_{nu}\left(Clo_{\eta}\left(Int_{\eta}\left(Clo_{\eta}(N)\right)\right)\right) & \widetilde{\subseteq} \\ Clo_{\lambda}\left(f_{nu}\left(Int_{\eta}\left(Clo_{\eta}(K)\right)\right)\right)\widetilde{\cap}f_{nu}\left(Clo_{\eta}\left(Int_{\eta}\left(Clo_{\eta}(N)\right)\right)\right) & = \\ & 0_{\mathcal{S}}. \end{array}$$

This ends the proof.

The following main result shows that the soft optimal map, which is soft s-open and soft pre-continuous, will preserve soft disjointness for the class of soft semi-open sets.

Theorem 10. Let $f_{nu}:(Z,\eta,\mathcal{T})\longrightarrow(Y,\lambda,\mathcal{S})$ be soft optimal, soft s-open, and soft pre-continuous. If $K,N\in SO(\eta)$, and $K\widetilde{\cap}N=0_{\mathcal{T}}$, then $f_{nu}(K)\widetilde{\cap}f_{nu}(N)=0_{\mathcal{S}}$.

Proof. Since $K,N\in SO(\eta)$, then $K\widetilde{\subseteq}Clo_{\eta}(Int_{\eta}(K))\widetilde{\subseteq}Clo_{\eta}\left(Int_{\eta}\left(Clo_{\eta}(K)\right)\right)$ and

$N\widetilde{\subseteq}Clo_{\eta}(Int_{\eta}(N))\widetilde{\subseteq}Clo_{\eta}\left(Int_{\eta}\left(Clo_{\eta}(N)\right)\right)$. Hence,

$f_{nu}(K)\widetilde{\subseteq}f_{nu}\left(Clo_{\eta}\left(Int_{\eta}\left(Clo_{\eta}(K)\right)\right)\right)$ and $f_{nu}(N)\widetilde{\subseteq}f_{nu}\left(Clo_{\eta}\left(Int_{\eta}\left(Clo_{\eta}(N)\right)\right)\right)$.

Therefore, by Proposition 6,

$$\begin{array}{rcl} f_{nu}(K)\widetilde{\cap}f_{nu}(N) & \widetilde{\subseteq} & f_{nu}\left(Clo_{\eta}\left(Int_{\eta}\left(Clo_{\eta}(K)\right)\right)\right)\widetilde{\cap}f_{nu}\left(Clo_{\eta}\left(Int_{\eta}\left(Clo_{\eta}(N)\right)\right)\right) \\ & = & 0_{\mathcal{S}}. \end{array}$$

3|Conclusion

The paper at hand deals with the introduction of the class of soft optimal maps and explains some of its fundamental features such as the rigorous inclusion of soft injective maps. We constructed some correlations between soft optimal maps and their analogs in general topology to clarify the structural linkages. With the conditions under which soft optimal maps preserve soft semi-Hausdorff and soft Hausdorff spaces, we have a better understanding of how the soft separation axioms behave in soft topological spaces.

Some probable directions of investigation for prospective research might be: (1) introducing soft optimally continuous maps, (2) introducing soft optimally irresolute maps, and (3) attempting an application of the soft optimal maps to a "decision-making problem" as in [55, 56].

Acknowledgments

The author would like to thank the referees for their valuable suggestions and helpful comments in improving this paper.

Author Contribution

The author has read and agreed to the published version of the manuscript.

Funding

The author declares that no external funding or support was received for the research presented in this paper, including administrative, technical, or in-kind contributions.

Data Availability

No data were used to support this paper.

Conflicts of Interest

The author declares no conflict of interest.

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